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RESEARCH ARTICLE

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Some Applications of Lawvere's Fixpoint Theorem

Abstract The famous diagonal argument plays a prominent role in set theory as well as in the proof of undecidability results in computability theory and incompleteness results in metamathematics. Lawvere (1969) brings to light the common schema among them through a pretty neat fixpoint theorem which generalizes the diagonal argument behind Cantor's theorem and characterizes self-reference explicitly in category theory. Not until Yanofsky (2003) rephrases Lawvere's fixpoint theorem using sets and functions, Lawvere's work has been overlooked by logicians. This paper will continue Yanofsky's work, and show more applications of Lawvere's fixpoint theorem to demonstrate the ubiquity of the theorem. For example, this paper will use it to construct uncomputable real number, unnameable real number, partial recursive but not potentially recursive function, Berry paradox, and fast growing Busy Beaver function. Many interesting lambda fixpoint combinators can also be fitted into this schema. Both Curry's Y combinator and Turing's Θ combinator follow from Lawvere's theorem, as well as their call-by-value versions. At last, it can be shown that the lambda calculus version of the fixpoint lemma also fits Lawvere's schema.

Keywords paradox, fixpoint, diagonalization, combinator

1 Introduction

In the late 19th century, Cantor created set theory and discovered that the cardi-

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nality of a powerset $P(A)$ is larger than the cardinality of A, which reveals that there are many different levels of infinity. Soon after that, the third fundamental crisis of mathematics was triggered by Russell paradox, which states that the class of all sets that are not members of themselves cannot be a set. The 1930s saw a series of negative results. Gödel proved that any consistent and strong enough formal axiomatic system cannot be complete, and no consistent strong enough formal axiomatic system can prove its own consistency. Tarski told us that the arithetical truth cannot be defined in arithmetic. Turing showed that the halting problem is unsolvable and the validity in first order logic is undecidable. These negative results shatters both Hilbert's program and Leibniz's great dream about the universal characteristic and rational calculus. It seems that such kind of paradoxed and theorems have already touched the boundary and limits of human reason. They even excited many great philosophers' interests. For example, paradoxes and theorems as above are full of deep philosophical implications according to Hofstadter (1979), and the famous diagonal methods are even involved in the arguments of the limits of artificial intelligence (Penrose 1999), so it is very important to study the common structures behind them. As is well known, all of the above famous negated results are related to the liar paradox—"I am lying"—where self-reference occurs. Lawvere (1969) brings to light the common schema among them through a pretty neat fixpoint theorem which generalizes the diagonal argument behind Cantor's theorem and characterizes self-reference explicitly in category theory. So concerning the great philosophical significance of Lawvere's fixpoint theorem, it is supposed to speak for itself. However, not until Yanofsky (2003) rephrases Lawvere's fixpoint theorem using sets and functions, Lawvere's paper has been overlooked by logicians and philosophers. Besides Yanofsky's examples, we will show more applications of Lawvere's fixpoint theorem in this paper to demonstrate the ubiquity of the abstract schema.

Notation. We assume the reader is familiar with the basics of lambda calculus and computability theory, which can be found in Odifreddi (1989). We write $[\![\varphi]\!]$ for the characteristic function of φ. $X \rightarrowtail Y$ is an injective map from X to Y, and $X \twoheadrightarrow Y$ is a surjective map from X to Y, and (f, g) maps x to $(f(x), g(x))$.

2 Lawvere's Fixpoint Theorem

Theorem 1 (Lawvere's Fixpoint Theorem). *In any category with finite products, if there is an object T and a morphism $f\colon T \times T \to Y$ such that, for every $g\colon T \to Y$ there is a $t\colon 1 \to T$ for all $x\colon 1 \to T$: $g \circ x = f \circ (x, t)$, then every endomorphism $\alpha\colon Y \to Y$ has a fixpoint $y\colon 1 \to Y$ such that $\alpha \circ y = y$.*

Yanofsky directly rephrases Lawvere's fixpoint theorem using sets and functions as follows.

Theorem 2 (Lawvere's Fixpoint Theorem—Yanofsky's Version). *If Y is a set and there exists a set T and a function $f\colon T \times T \to Y$ such that all functions $g\colon T \to Y$ are representable by f (there exists a $t \in T$ such that $g(-) = f(-, t)$) then all functions $\alpha\colon Y \to Y$ have a fixpoint.*

$$\begin{tikzcd}
T \times T \arrow[r, "f"] & Y \arrow[d, "\alpha"] \\
T \arrow[u, "(Id, Id)"] \arrow[r, "g"'] & Y
\end{tikzcd}$$

Yanofsky remarks that:

> Obviously, every set Y with two or more elements has a function to itself that does not have a fixed point. It is here that we get in trouble for talking about sets and functions as opposed to objects in a category and morphisms between those objects. Perhaps Y and T are sets with extra (algebraic) structure and functions between them are intended to preserve that extra structure. In that case, we are really dealing with fewer functions between the sets. (Yanofsky 2003, 366)

For Yanofsky's version, due to the limit of the language of sets and functions, something relevant is missing. But we should not blame set theory for the trouble. Any function $f\colon X \to Z^Y$ corresponds to a function $\hat{f}\colon X \times Y \to Z$ where $\hat{f}(x, y) = f(x)(y)$ for $x \in X$ and $y \in Y$. Conversely, any function $f\colon X \times Y \to Z$ corresponds to a function $\hat{f}\colon X \to Z^Y$ where $\hat{f}(x)(y) = f(x, y)$. It is easy to see that all functions $g\colon T \to Y$ are representable by $f\colon T \times T \to Y$ if and only if there exists $f\colon T \twoheadrightarrow Y^T$. Yanofsky's version of Lawvere's theorem 2 can be equivalently rephrased as "*if there exists $f\colon X \twoheadrightarrow Y^X$, then every $\alpha\colon Y \to Y$ has a fixpoint.*" However, we do not need that all functions $g\colon T \to Y$ are representable by f. We only need all functions of the form $g = \alpha \circ f \circ (Id, \beta)$ be representable by f. Lawvere's fixpoint theorem is stronger than Yanofsky's version. Lawvere's fixpoint theorem can be reformulated more general.

Definition 1 (Representability).
A function $g\colon X \to Z$ is representable *by $f\colon X \times Y \to Z$ if and only if $\exists y \in Y \forall x \in X(g(x) = f(x, y))$.*

Theorem 3 (Lawvere's Fixpoint Theorem). *For sets X, Y, Z, and functions $\beta\colon X \to Y, f\colon X \times Y \to Z, \alpha\colon Z \to Z$, let $g := \alpha \circ f \circ (Id, \beta)$, assume β is surjective,*

(I) if α has no fixpoint, then g is not representable by f.

(II) if g is representable by f, then α has a fixpoint.

$$\begin{tikzcd}
X\times Y \arrow[r, "f"] & Z \arrow[d, "\alpha"] \\
X \arrow[u, "{(Id,\beta)}"] \arrow[r, "g"'] & Z
\end{tikzcd}$$

The proof is easy. For (II), if $g = \alpha \circ f \circ (Id, \beta)$ is representable by t, then it is easy to check that $f(\beta^{-1}(t), t)$ is a fixpoint of α. And (I) is the contrapositive of (II).

Since the functions f, g, α are quantified, α is restricted by the prerequisite. It is not that all functions α must have a fixpoint, but that all functions α which can make functions $\alpha \circ f \circ (Id, \beta)$ representable by f must have a fixpoint. Yanofsky's version 2 is quite weak, and the above one is more general.

In the following sections we show some applications of Lawvere's fixpoint theorem. Many interesting paradoxes, Cantor's theorem, fast growing function, partial recursive but not potentially recursive function, and Turing's halting theorem are all consequences of Lawvere's fixpoint theorem 3-(I), while many interesting fixpoint theorems, fixpoint lemma and fixpoint combinators are instances of Lawvere's fixpoint theorem 3-(II).

3 Applications of Lawvere's Theorem in Set Theory

Russell-like Paradox. Grelling paradox asks: Is "non-self-descriptive" non-self-descriptive? By the following graph, we can see that "non-self-descriptive" is not representable.

$$\begin{tikzcd}
X\times X \arrow[r, "f"] & 2 \arrow[d, "\alpha"] \\
X \arrow[u, "(Id, Id)"] \arrow[r, "g"'] & 2
\end{tikzcd}$$

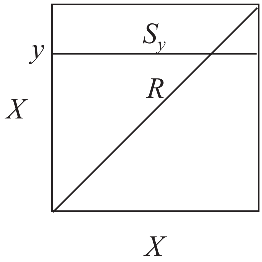
where $f\colon (x, y) \mapsto [\![y \text{ "describes" } x]\!]$ and $\alpha\colon x \mapsto 1 - x$.

If we let $X :=$ "the universe of all sets," $f\colon (x, y) \mapsto [\![x \in y]\!]$, and $\alpha\colon x \mapsto 1-x$, then we get the famous Russell paradox. The function $g(x) = [\![x \notin x]\!]$ can be represented by the Russell "set" $R := \{x\colon x \notin x\}$ if it exists, namely, for all $x\colon g(x) = f(x, R)$. According to Lawvere's theorem, α has a fixpoint, and $f(R, R)$ is its fixpoint, which leads to the contradiction.

The Barber paradox is an applied version of Russell paradox. If we let X :="men in the village," $f\colon (x, y) \mapsto [\![y \text{ "shaves" } x]\!]$, and $\alpha\colon x \mapsto 1 - x$, then we get the Barber paradox.

Generally, let $S \subset X \times X$. The relation S may be read as "shave" or "membership" or "describe" or any other binary relation on X. As depicted in Figure 1, let $S_y := \{x\colon S(x, y)\}$, and $R := \{x\colon x \notin S_x\}$, then R is not representable by S: $\forall x(R \neq S_x)$.

$$\begin{tikzcd} X\times X \arrow[r, "f"] & 2 \arrow[d, "\alpha"] \\ X \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & 2 \end{tikzcd}$$



where $f\colon (x,y) \mapsto [\![S(x,y)]\!]$ and $\alpha\colon x \mapsto 1-x$.

Figure 1 Russell-Like Paradox

The liar paradox, Quine paradox and Richardian number are similar.

To prove the set of real numbers is uncountable, Cantor constructed a new real that does not occur in the assumed representation by the diagonal argument. It can be reproduced through Lawvere's theorem as follows.

Theorem 4 (Cantor). *$\mathbb{R}$ is uncountable.*

Proof.

$$\begin{tikzcd} \mathbb{N}\times\mathbb{N} \arrow[r, "f"] & 10 \arrow[d, "\alpha"] \\ \mathbb{N} \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & 10 \end{tikzcd}$$

where $10 := \{0,1,2\ldots,9\}$, and $f\colon (m,n) \mapsto r_{mn} :=$ "the n^{th} digit of the m^{th} real," and $\alpha\colon x \mapsto 9-x$. Suppose f is an enumeration of $\mathbb{R}$, we construct a new real $\sum_n g(n)10^{-n}$ which does not occur in the enumeration, therefore $\mathbb{R}$ is uncountable. $\square$

Since the set of Turing machines is countable, it follows directly that there exists uncomputable reals—reals that cannot be calculated digit by digit by any Turing machine. We can also prove it in a constructive way. If we let

$$f\colon (m,n) \mapsto r_{mn} := \begin{cases} \text{the } n^{th} \text{ digit output by the } m^{th} \text{ Turing machine} \\ 0 \ \ \text{if the } m^{th} \text{ Turing machine never outputs a } n^{th} \text{ digit,} \end{cases}$$

then the real $\sum_n g(n)10^{-n}$ would diagonalize over all the computable reals and immediately yield an uncomputable real.

As is well known Richard's paradox has something to do with Cantor's diagonal argument. It is not hard to see that Richard's paradox can be reformulated in exactly the same way by defining $f\colon (m,n) \mapsto r_{mn}$:="the n^{th} digit of the real number named by the m^{th} sentence."

Not only the cardinality of $\mathbb{R}$ is larger than the cardinality of $\mathbb{N}$, but for any set X, the cardinality of its power set is larger than itself. We can prove Cantor's theorem in three different ways, but with the same schema. Three ways to prove Cantor's theorem.

Theorem 5 (Cantor's Theorem).

$$|X| < |P(X)|$$

Proof.

$$\begin{tikzcd}
X\times P(X) \arrow[r, "f"] & 2 \arrow[d, "\alpha"] \\
X \arrow[u, "{(Id,\beta)}"] \arrow[r, "g"'] & 2
\end{tikzcd}$$

where $f\colon (x,y) \mapsto [\![x \in y]\!]$ and $\alpha\colon x \mapsto 1 - x$. For any β, α is fixpoint free, and every g is representable by f, so β is not surjective, in other words, by the contraposition of Lawvere's theorem, we get $|X| < |P(X)|$. □

Proof. Assume $h\colon P(X) \rightarrowtail X$.

$$\begin{tikzcd}
P(X)\times P(X) \arrow[r, "f"] & 2 \arrow[d, "\alpha"] \\
P(X) \arrow[u, "{(Id,Id)}"] \arrow[r, "g"'] & 2
\end{tikzcd}$$

where $f\colon (x, y) \mapsto [\![h(x) \in y]\!]$, and $\alpha\colon x \mapsto 1 - x$. It is not hard to check that g is representable by $y := \{h(x)\colon x \subset X \ \&\ h(x) \notin x\}$. Contradiction! $\square$

Proof. If $|X| \geqslant |P(X)|$, then there exists some X-enumeration $\{S_i\}_{i\in X}$ of $P(X)$.

$$\begin{tikzcd}
X\times\{S_i\}_{i\in X} \arrow[r, "f"] & 2 \arrow[d, "\alpha"] \\
X \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & 2
\end{tikzcd}$$

where $f\colon (x, y) \mapsto [\![x \in S_y]\!]$ and $\alpha\colon x \mapsto 1 - x$. Then $g(x) = [\![x \notin S_x]\!]$. Since $\{S_i\}_{i\in X}$ is the enumeration of $P(X)$, the set $R := \{x\colon x \notin S_x\}$ that g characterizes must be some S_t: $\exists t(R = S_t)$. It means g is representable by t. Contradiction! $\square$

From the last proof, we see just the same structure as Russell paradox (Figure 1). It shows that there are many different levels of infinity. With Cantor's theorem, a even more amazing result is not hard to prove: the "set" of all distinct levels of infinity is so large that it cannot even be called a set!

We have proved Cantor's theorem in three different ways, using three different commuting graphs, but with the same Lawvere schema. All of them involve the "diagonalization" process (Id, Id), the "evaluation" process f and the "negation" process α.

Actually, Lawvere's theorem is a reformulation of the diagonalization trick that is at the heart of Cantor's theorem. Since $|P(X)| = |2^X|$, Cantor's theorem says $|X| < |2^X|$.

Theorem 6 (Cantor). *For* $|Y| \geqslant 2$,

$$|X| < |Y^X|$$

Proof.

$$\begin{tikzcd} X{\times}X \arrow[r, "f"] & Y \arrow[d, "\alpha"] \\ X \arrow[u, "(Id, Id)"] \arrow[r, "g"'] & Y \end{tikzcd}$$

where α is the cyclic permutation. Since every $g\colon X \to Y$ is representable by some $f\colon X{\times}X \to Y$ if and only if there exists $f\colon X \twoheadrightarrow Y^X$, the unrepresentability of g implies that there is no surjective map from X to Y^X. $\square$

We know that Yanofsky's version of Lawvere's theorem 2 can be rephrased as "*if there exists $f\colon X \twoheadrightarrow Y^X$, then every $\alpha\colon Y \to Y$ has a fixpoint*." However, as Yanofsky remarks, this version of Lawvere's theorem falls in trouble. Cantor's theorem shows that X is never big enough to represent all the maps $X \to Y$ by some $f\colon X \times X \to Y$, so the premise $X \twoheadrightarrow Y^X$ is always false when $|Y| \geqslant 2$, and false premise implies everything. In this sense it seems that Lawvere's theorem is a trivial. However, we only need g of the form $\alpha \circ f \circ (Id, \beta)$ to be representable by f rather than $X \twoheadrightarrow Y^X$, so Lawvere's theorem is stronger than Yanofsky's version.

Even if the above version of Lawvere's fixpoint theorem has nontrivial applications. For example, most spaces admit fixpoint free function, then it follows that, for most spaces, there is no space-filling curve for its path space. Since a continuous function on $\mathbb{R}$ is determined by its values at rational points, the set $C(\mathbb{R}, \mathbb{R})$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ has the same cardinality as $\mathbb{R}$. But is there a continuous surjection $\mathbb{R} \twoheadrightarrow C(\mathbb{R}, \mathbb{R})$ from the real line to the Banach space of continuous real functions, equipped with the sup-norm $||f||_\infty = \sup\limits_{x \in \mathbb{R}} |f(x)|$?

$$\begin{tikzcd} \mathbb{R}{\times}C(\mathbb{R},\mathbb{R}) \arrow[r, "\mathscr{F}"] & \mathbb{R} \arrow[d, "\alpha"] \\ \mathbb{R} \arrow[u, "(Id, \beta)"] \arrow[r, "g"'] & \mathbb{R} \end{tikzcd}$$

where $\mathscr{F}\colon (x, f) \mapsto f(x)$ and $\alpha\colon x \mapsto x + 1$. The negative answer follows from Lawvere's theorem because α is continuous and has no fixpoint.

4 Applications of Lawvere's Theorem in Computability Theory

In the following we show some applications of Lawvere's theorem in computability theory.

Theorem 7. *There exists total recursive but not primitive recursive functions.*

Proof.

$$\begin{tikzcd}
\mathbb{N}\times\mathbb{N} \arrow[r, "f"] & \mathbb{N} \arrow[d, "\alpha"] \\
\mathbb{N} \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & \mathbb{N}
\end{tikzcd}$$

where $f\colon (m, n) \mapsto \psi_n(m)$, and $\alpha\colon x \mapsto x + 1$. Then $g(n) = \psi_n(n) + 1$. The function f is an enumeration of all the primitive recursive functions. It is total recursive but not primitive recursive. Since α is fixpoint free, if f is total primitive recursive, then g is also total primitive recursive, but g diagonalizes all of the primitive recursive functions and is not representable by f. □

Although f is not primitive, it is total recursive. The set of primitive recursive functions can be effectively enumerated. But is there any effective enumeration of the total recursive functions? The answer is negative. Since if there exists such an enumeration f, then a counterexample g can be constructed in the same way as above to transcend the predetermined enumeration.

If we want a total recursive but not primitive recursive function which can dominate all of the primitive recursive functions, we can let $f\colon (m, n) \mapsto \max\limits_{k\leqslant n} \psi_k$ (m). Then $g(m) = \max\limits_{k\leqslant m} \psi_k(m) + 1$. Obviously, g is not representable, and for any n, the inequality $g(m) > \psi_n(m)$ holds for all but finitely many m.

In the above argument, we construct some counterexample that diagonalizes the presupposed enumeration. Similarly, we can define a busy beaver function that dominates all of the partial recursive functions. Let $f\colon (m,n) \mapsto \max\limits_{k\leqslant n} \varphi_k(m)$. Then $g(m) = \max\limits_{k\leqslant m} \varphi_k(m) + 1 = \max\{n\colon K(n|m) \leqslant m\} + 1$, where $K(n|m) := \mu e[\varphi_e(m) = n]$, and $K(n) := K(n|0)$. Since α is fixpoint free, the function g is unrepresentable. Because for any n, $K(\varphi_n(m)|m) = K(\varphi_n) < m$, we can see that $g(m) > \varphi_n(m)$ for almost all m. Usually the busy beaver function is defined by $BB(m) := \max\{\varphi_k(0)\colon k \leqslant m\} = \max\{n\colon K(n) \leqslant m\}$. Now similar to BB, the function g is also a busy beaver function that dominates all of the partial recursive functions.

Since the Berry paradox has something to do with the busy beaver function, we can use the Berry paradox to diagonalize all of the partial recursive functions.

$$
\begin{tikzcd}
\mathbb{N}\times\mathbb{N} \arrow[r, "\mathscr{E}_\varphi"] & \{\varphi_n(m)\}_{(m,n)\in\mathbb{N}^2} \arrow[d, "\alpha"] \\
\mathbb{N} \arrow[u, "(Id, Id)"] \arrow[r, "g"'] & \{\varphi_n(m)\}_{(m,n)\in\mathbb{N}^2}
\end{tikzcd}
$$

where $\mathscr{E}_\varphi\colon (m,n) \mapsto \varphi_n(m)$, and $\alpha\colon \varphi_n(m) \mapsto \min(\mathbb{N} \setminus \{\varphi_k(m)\colon k \leqslant n\})$. Obviously, α is fixpoint free, and $g(m) = \min(\mathbb{N}\setminus\{\varphi_k(m)\colon k \leqslant m\}) = \mu n\,[K(n|m) > m]$. Intuitively, the function g is a characterization of Berry paradox—"the least number undefinable in fewer than ten words." The number $g(m)$ is the least one that cannot be computed with programs whose code is not larger than m, even given m as input. And g can be seen as sort of counterpart of the busy beaver function. Since α is fixpoint free, the function g is unrepresentable. It follows that g is uncomputable. It also implies the uncomputability of the function K, otherwise, g would be computable.

In the following we show that not every partial recursive function is potentially recursive.

Definition 2 . *The function* $\overline{f}$ *is a* completion *of a partial function* f *if* $\overline{f}$

is total and $\forall n\colon f(n)\downarrow \;\Rightarrow\; f(n) = \overline{f}(n)$. *A partial function* f *is* potentially recursive *if it has a completion which is recursive.*

Theorem 8 . *Not every partial recursive function is potentially recursive.*

Proof.

$$\begin{tikzcd}
\mathbb{N}\times\mathbb{N} \arrow[r, "\mathscr{E}\varphi"] & \{\varphi_n(m)\}_{(m,n)\in\mathbb{N}^2} \arrow[d, "\alpha"] \\
\mathbb{N} \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & \{\varphi_n(m)\}_{(m,n)\in\mathbb{N}^2}
\end{tikzcd}$$

where $\mathscr{E}_\varphi\colon (m,n) \mapsto \varphi_n(m)$, and $\alpha\colon x \mapsto x+1$. Then $g(m) = \varphi_m(m)+1$. Obviously, g partial recursive $\Rightarrow$ g representable $\Rightarrow$ $\alpha(g(\ulcorner g\urcorner)) = g(\ulcorner g\urcorner)$, but α is fixpoint free, so $g(\ulcorner g\urcorner)\uparrow$. Now we show that for any partial recursive $\overline{g}\supset g\colon \overline{g}(\ulcorner\overline{g}\urcorner)\uparrow$.

$$\overline{g}(\ulcorner\overline{g}\urcorner) = \varphi_{\ulcorner\overline{g}\urcorner}(\ulcorner\overline{g}\urcorner) = g(\ulcorner\overline{g}\urcorner) = \varphi_{\ulcorner\overline{g}\urcorner}(\ulcorner\overline{g}\urcorner)+1. \quad \text{Contradiction!} \qquad \square$$

The unsolvability of Turing's halting problem also follows from Lawvere's fixpoint theorem.

Theorem 9 (Turing). *The Halting problem is unsolvable.*

Proof.

$$\begin{tikzcd}
\mathbb{N}\times\mathbb{N} \arrow[r, "H"] & 2 \arrow[d, "\alpha"] \\
\mathbb{N} \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & 2
\end{tikzcd}$$

where $H\colon(x,y)\mapsto \llbracket \varphi_y(x)\downarrow\rrbracket$, and $\alpha(x)=\begin{cases}1 & \text{if } x=0\\ \uparrow & \text{otherwise}\end{cases}$ is fixpoint free. If H is recursive, then the function g constructed as above is partial recursive. Hence $g=\varphi_t$ for some t. It is not hard to see that $H(t,t)\uparrow$. □

The above proof is formalized in Yanofsky (2003). Here we give another proof of theorem 9 that can also be fitted into the schema of Lawvere's theorem.

Proof.

$$\begin{tikzcd}
\mathbb{N}\times\mathbb{N} \arrow[r, "\mathscr{E}_\varphi"] & \{\varphi_n(m)\}_{(m,n)\in\mathbb{N}^2} \arrow[d, "\alpha"] \\
\mathbb{N} \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & \{\varphi_n(m)\}_{(m,n)\in\mathbb{N}^2}
\end{tikzcd}$$

where $\mathscr{E}_\varphi\colon(m,n)\mapsto\varphi_n(m)$, and $\alpha\colon\varphi_n(m)\mapsto 1+\sum_{k=0}^{n}H(m,k)\cdot\varphi_k(m)$. If H is total recursive, then the function g constructed as above is also total recursive. Since α is fixpoint free, g is unrepresentable. But it means g is uncomputable. Contradiction! □

Fixpoint. Lawvere's fixpoint theorem 3-(II) also has many interesting applications. Kleene's fixpoint theorem, fixpoint lemma in logic, and many important fixpoint combinators in lambda calculus all follow from Lawvere's fixpoint theorem 3-(II).

Yanofsky (2003) shows how to fit Kleene's fixpoint theorem and fixpoint lemma into Lawvere's schema. In order to make comparison, we copy the two results as follows, and prove some new consequences of Lawvere's theorem in the next section. All of them are of the same schema.

Theorem 10 (Kleene's Fixpoint Theorem). *Given a total recursive function h, there is an $e\in\mathbb{N}$ s.t.*

$$\varphi_e=\varphi_{h(e)}$$

Proof.

$$\begin{tikzcd} \mathbb{N}\times\mathbb{N} \arrow[r, "f"] & \{\varphi_n\}_{n\in\mathbb{N}} \arrow[d, "\mathscr{E}_h"] \\ \mathbb{N} \arrow[u, "(Id, Id)"] \arrow[r, "g"'] & \{\varphi_n\}_{n\in\mathbb{N}} \end{tikzcd}$$

where $f\colon (m,n) \mapsto \varphi_{\varphi_n(m)}$, and $\mathscr{E}_h\colon \varphi_n \mapsto \varphi_{h(n)}$. The function $g = \left(\varphi_{h(\varphi_n(n))}\right)_{n\in\mathbb{N}}$ is a recursive sequence of partial recursive functions, and thus is representable by f. Then a fixpoint for $\mathscr{E}_h$ exists, and we can explicitly construct it: $g(m) = \varphi_{h(\varphi_m(m))} = \varphi_{s(m)} = \varphi_{\varphi_t(m)} = f(m,t)$. Obviously, $e := \varphi_t(t)$ is the fixpoint. □

By Kleene's fixpoint theorem, von Neumann's self-reproducing machine and totally introspective program can be constructed.

Theorem 11 (Fixpoint Lemma). *Let Q be Robinson arithmetic. For any formula $\alpha(x)$ with just one free variable x, there exists a sentence β s.t.*

$$Q \vdash \beta \leftrightarrow \alpha\left(\ulcorner\beta\urcorner\right)$$

Proof.

$$\begin{tikzcd} \mathrm{Lin}_1\times\mathrm{Lin}_1 \arrow[r, "f"] & \mathrm{Lin}_0 \arrow[d, "\mathscr{E}_\alpha"] \\ \mathrm{Lin}_1 \arrow[u, "(Id, Id)"] \arrow[r, "g"'] & \mathrm{Lin}_0 \end{tikzcd}$$

where Lin_1 is the set of formulas with only one free variable, and Lin_0 is the set of sentences. The function $f\colon (\varphi(x), \psi(x)) \mapsto \psi\left(\ulcorner\varphi(x)\urcorner\right)$, and $\mathscr{E}_\alpha\colon \varphi \mapsto \alpha(\ulcorner\varphi\urcorner)$. Then $g(\varphi(x)) = \alpha\left(\ulcorner\varphi\left(\ulcorner\varphi(x)\urcorner\right)\urcorner\right)$. If we let $\gamma(x) \equiv \alpha\left(D(x)\right)$, where $D\colon \ulcorner\varphi(x)\urcorner \mapsto \ulcorner\varphi\left(\ulcorner\varphi(x)\urcorner\right)\urcorner$, then $\beta \equiv \gamma\left(\ulcorner\gamma(x)\urcorner\right)$ is the fixpoint. □

Although Gödel's first incompleteness theorem can be proven in various ways, for example, it could be proven from the Berry paradox, or from the busy beaver functions, or from Kleene's normal form theorem and the existence of partial recursive but not potentially recursive function 8, or from the unsolvability of Turing's halting problem 9. However, the fixpoint lemma 11 plays a central role in the proof of most of the incompleteness results. Gödel's first incompleteness theorem, Gödel-Rosser's incompleteness theorem, Tarski's undefinability of truth theorem, Löb's Theorem, Parikh's theorem all follow from the fixpoint lemma. It seems that the construction of the fixpoint in the fixpoint lemma is more complicated than the fixpoint in Kleene's fixpoint theorem at the first sight. But if we compare the total recursive function h in Kleene's theorem with the formula α in fixpoint lemma, and compare the application function f in Kleene's theorem with the application function f in fixpoint lemma, we can find similar construction processes. We show that the very same process also gives us a fixpoint combinator in lambda calculus—Curry's Y combinator.

5 Applications of Lawvere's Theorem in Lambda Calculus

Assume Λ is the set of lambda terms.

$$\begin{tikzcd}
\Lambda\times\Lambda \arrow[r, "f"] & \Lambda \arrow[d, "\mathscr{E}_y"] \\
\Lambda \arrow[u, "(Id, Id)"] \arrow[r, "g"'] & \Lambda
\end{tikzcd}$$

where $f\colon (x, y) \mapsto yx$, and $\mathscr{E}_y\colon x \mapsto yx$. Then

$$g = \lambda x.y(xx).$$

By applying g to itself we then have a fixpoint of y

$$gg = y(gg).$$

We can abstract y from g and obtain the Y combinator

$$Y := \lambda y.gg = \lambda y.(\lambda x.y(xx))(\lambda x.y(xx)).$$

By applying Y to any term φ we get

$$Y\varphi = \varphi(Y\varphi) = \varphi(\varphi(Y\varphi)) = \cdots$$

Actually, Kleene's fixpoint theorem10 follows from the call-by-value version of Y combinator—the Z combinator, which also fits Lawvere's schema.

$$\begin{tikzcd}
\Lambda\times\Lambda \arrow[r, "f"] & \Lambda \arrow[d, "\mathscr{E}y"] \\
\Lambda \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & \Lambda
\end{tikzcd}$$

where $f\colon (x, y) \mapsto \lambda v.yxv$, and $\mathscr{E}_y\colon x \mapsto yx$. Then

$$g = \lambda x.y(\lambda v.xxv).$$

Just like the Y combinator, we can define

$$Z := \lambda y.gg = \lambda y.(\lambda x.y(\lambda v.xxv))(\lambda x.y(\lambda v.xxv)).$$

By applying the Z combinator to h we get

$$Zhv = h(Zh)v.$$

If we let $e := Zh$ then we obtain the lambda calculus version of the Kleene's fixpoint theorem 10 $ev = hev$.

Not only the Y, Z combinator follows from Lawvere's theorem, so does Turing's Θ combinator, as well as the call-by-value Θ_v combinator.

$$\begin{tikzcd}
\Lambda\times\Lambda \arrow[r, "f"] & \Lambda \arrow[d, "\alpha"] \\
\Lambda \arrow[u, "{(Id, Id)}"] \arrow[r, "g"'] & \Lambda
\end{tikzcd}$$

where $f\colon (x, y) \mapsto yx$, and $\alpha\colon x \mapsto \lambda y.y(xy)$. Then

$$g = \lambda xy.y(xxy).$$

We can define the Θ combinator as follows

$$\Theta := gg = (\lambda xy.y(xxy))(\lambda xy.y(xxy)).$$

It can be used just like the Y combinator.

$$\Theta\varphi = \varphi(\Theta\varphi) = \varphi(\varphi(\Theta\varphi)) = \cdots$$

Now we prove that the call-by-value Θ_v combinator also follows from Lawvere's fixpoint theorem.

$$\begin{tikzcd}
\Lambda\times\Lambda \arrow[r, "f"] & \Lambda \arrow[d, "\alpha"] \\
\Lambda \arrow[u, "(Id, Id)"] \arrow[r, "g"'] & \Lambda
\end{tikzcd}$$

where $f\colon (x, y) \mapsto yx$, and $\alpha\colon x \mapsto \lambda y.y(\lambda z.xyz)$. Then

$$g = \lambda xy.y(\lambda z.xxyz).$$

The call-by-value Θ_v combinator can be defined as

$$\Theta_v := gg = (\lambda xy.y(\lambda z.xxyz))(\lambda xy.y(\lambda z.xxyz)).$$

And it works just like the Z combinator

$$\Theta_v hv = h(\Theta_v h)v.$$

So the Θ_v combinator can also be used to prove the lambda calculus version of the Kleene's fixpoint theorem if we let $e := \Theta_v h$.

At last, let is prove the lambda calculus version of the fixpoint lemma 11.

Theorem 12 (Fixpoint Lemma—Lambda Calculus Version). *For every λ-term F there is a λ-term X s.t.*

$$F\ulcorner X\urcorner = X.$$

Proof.

$$\begin{tikzcd}
\underline{\Lambda}\times\underline{\Lambda} \arrow[r, "A"] & \Lambda \arrow[d, "\mathscr{E}F"] \\
\underline{\Lambda} \arrow[u, "(Id,Id)"] \arrow[r, "G"'] & \Lambda
\end{tikzcd}$$

where $\underline{\Lambda} := \{\ulcorner M\urcorner : M \in \Lambda\}$, and $A : (\ulcorner M\urcorner, \ulcorner N\urcorner) \mapsto N(\ulcorner M\urcorner)$, and $\mathscr{E}_F : M \mapsto F\ulcorner M\urcorner$.

$$G\ulcorner M\urcorner = F\ulcorner M\ulcorner M\urcorner\urcorner.$$

Then $X := G\ulcorner G\urcorner$ is the fixpoint. □

6 Conclusion

Lawvere's fixpoint theorem is a generalization of the diagonal method which is widely used in the construction of paradoxes—such as the liar paradox, Grelling paradox, Russell paradox, Barber paradox, Richard paradox, and in the proof of many important theorems in set theory and computability theory—such as Cantor's theorem, Gödel's first incompleteness theorem, Gödel-Rosser's incompleteness theorem, Tarski's undefinability of truth theorem, Löb's theorem, Parikh's theorem, Kleene's fixpoint theorem, Turing's halting theorem, Rice's theorem, von Neumann's self-reproducing automata, and totally introspective program. In this paper, we express Lawvere's theorem using sets and functions more general than Yanofsky's version, and show more applications of it to demonstrate the ubiquity of the theorem. For example, we use it to construct uncomputable real number, unnameable real number, partial recursive but not potentially recursive function, Berry paradox, and fast growing functions that diagonalize the class of primitive recursive functions and partial recursive functions. Many interesting lambda fixpoint combinators can also be fitted into this schema. Both Curry's Y combinator and Turing's Θ combinator follow from Lawvere's theorem, as well as their call-by-value versions—the Z combinator and Θ_v combinator, from which Kleene's fixpoint theorem can be seen as a corollary. At last, we show that the lambda calculus version of the fixpoint lemma also fits Lawvere's schema.

As shown in Table 1, compare the Y combinator with the lambda calculus version of fixpoint lemma, the logic version of fixpoint lemma, Kleene's fixpoint theorem and Russell paradox, we can see the similarity between one another.

Table 1 Fixpoint and Diagonalization

Y combinator	$\hat{=}$	Fixpoint 12	$\hat{=}$	Fixpoint 11	$\hat{=}$	Fixpoint 10	$\hat{=}$	Russell Paradox
yx	$\hat{=}$	$N(\ulcorner M \urcorner)$	$\hat{=}$	$\psi(\ulcorner \varphi(x) \urcorner)$	$\hat{=}$	$\varphi_n(m)$	$\hat{=}$	$x \in y$
xx	$\hat{=}$	$M(\ulcorner M \urcorner)$	$\hat{=}$	$\varphi(\ulcorner \varphi(x) \urcorner)$	$\hat{=}$	$\varphi_n(n)$	$\hat{=}$	$x \in x$
$y(xx)$	$\hat{=}$	$F \ulcorner M \ulcorner M \urcorner \urcorner$	$\hat{=}$	$\alpha(\ulcorner \varphi(\ulcorner \varphi(x) \urcorner) \urcorner)$	$\hat{=}$	$h(\varphi_n(n))$	$\hat{=}$	$x \notin x$
$\lambda x.y(xx)$	$\hat{=}$	G	$\hat{=}$	$\gamma(x)$	$\hat{=}$	$\varphi_t(n)$	$\hat{=}$	$x \notin R$
$(\lambda x.y(xx))(\lambda x.y(xx))$	$\hat{=}$	$G(\ulcorner G \urcorner)$	$\hat{=}$	$\gamma(\ulcorner \gamma(x) \urcorner)$	$\hat{=}$	$\varphi_t(t)$	$\hat{=}$	$R \notin R$

Review Lawvere's fixpoint theorem 3, except that Yanofsky's version is weaker, theorem 3-(I) corresponds to Cantor's theorem in Yanofsky (2003, 366), while theorem 3-(II) corresponds to diagonal theorem in Yanofsky (2003, 377). The map (Id, β) generalizes (Id, Id), and it formally characterizes "diagonalization." Z can be seen as the set of "truth values," and f is kind of "evaluation" or "application" function, so $f(t,t)$ can be read that t is an evaluation of itself. The essence of the diagonal method is the fact of using one item t on two different levels—the object-level X and the meta-level Y. The meta-level Y is the instrument that we can use to represent all the maps (properties) $X \to Z$ with $f\colon X \times Y \to Z$.

On one hand, if g is representable by $f(-,t)$, then $g(\beta^{-1}(t))$ is indirect self-application, and $f(\beta^{-1}(t),t)$ is a fixpoint of α. With fixpoint, the system has kind of "*self-reflexivity*." The fixpoint $f(\beta^{-1}(t),t)$ says that—"I have the property α." For example, in Tarski's undefinability of truth theorem, its fixpoint says "I am not true"; in Gödel's first incompleteness theorem, its fixpoint says "I am not provable"; in Gödel-Rosser's incompleteness theorem, its fixpoint says "for every proof of me, there is a shorter proof of my negation"; in Löb's theorem, its fixpoint says "if I am provable, then φ" for any sentence φ. And this fixpoint is related to Curry's paradox: "if this sentence is true, then Santa Claus exists"; in Parikh's theorem, its fixpoint says "I have no proof of myself shorter than n."

On the other hand, if α—the generalization of "negation"—has no fixpoint, then $f(\beta^{-1}(t), t)$ is "illegitimate," which means we construct some new item $g(\beta^{-1}(t))$ by "negation" and self-reference that "*transcends*" the predetermined list represented by Y. The existence of uncomputable real number, unnameable real number, partial recursive but not potentially recursive function, and fast growing Busy Beaver functions are instances of such kind of construction.

Perhaps both "*self-reflexivity*" and "*self-transcendence*" would play central roles in the philosophy of mind or in the design of artificial intelligence. Thanks to Lawvere's fixpoint theorem, we not only see the ubiquity of the diagonalization schema and the profound analogies between different theorems, proofs and paradoxes, but also see the deep trade-off between "*self-reflexivity*" and "*self-transcendence*."

Future Research. Yanofsky hopes that Ackermann function, Paris-Harrington theorem, Kripke's truth theory, Brouwer's fixpoint theorem, Nash's equilibrium theorem, Tarski's fixpoint theorem, and Chaitin's incompleteness theorem, Gödel's second incompleteness theorem, and even Gödel's completeness theorem can be fitted into Lawvere's schema, but it seems quite difficult to fit them all in the same schema nontrivially, because the proof techniques of them are quite different (Yanofsky 2003, 383). However, the incompressibility method is at the heart of all Chaitin-like incompleteness theorems, is there a way to generalize the incompressibility method as succinct as Lawvere's fixpoint theorem?

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